Int. J. Solids Structures Vol. 24, No. 6, pp. 619–636, 1988
Printed in Great Britain.

0020-7683/88 $3.00+.00

A HYPERCIRCLE METHOD OF FRAME ANALYSIS—I. THEORY

C. A. Nelson
Honeywell Inc., 7225 Northland Drive, Brooklyn Park, MN 55428, U.S.A.

and

L. E. Goodman†
Civil and Mineral Engineering Department, University of Minnesota, Minneapolis, MN 55455, U.S.A.

(*Received 14 January 1987; in revised form 13 November 1987*)

Abstract—The hypercircle method of Synge and Prager is applied to the structural analysis of rigid-jointed planar frames. In Part I theoretical foundations of the method are developed and bounding formulae derived. These provide upper and lower bounds, as narrow as desired, on strain energy, local deformation and internal stress. A new procedure for reducing the number of statical and kinematical degrees of freedom is introduced.

INTRODUCTION

In 1947 Prager and Synge[1] introduced a new method of elastic stress analysis based on the concept of function space. In this method a function (e.g. the stress field in an elastic body) is regarded as a vector. Once a suitable inner product has been defined in the space of these vectors they may be treated in accordance with the mathematical rules for inner product spaces. Synge and Prager chose the integrated combination of a compatible strain field and an equilibrium stress field as the inner product and showed that the compatibility and equilibrium equations of linear elasticity define orthogonal hyperplanes with respect to this product. Guided by the familiar geometry of Euclidean space they developed the concepts of the "hypersphere" and "hypercircle" and used them to derive bounds on the strain energy of the solutions to boundary-value problems in linear elasticity. In a postscript they also derived a formula that could be used to obtain bounds on pointwise quantities such as stress and deformation. Synge's 1957 monograph[2] provides a readable account of the hypercircle method.

The application of the hypercircle method to structural analysis was treated by Prager[3] in 1950. In Ref. [3] Prager dealt with trusses and, briefly, with beams in flexure. The analysis of rigid-jointed frames without sidesway was cited by Prager as a problem to which the method could be applied. It was not until 1977, however, that the application of the hypercircle method to frame analysis was again considered. Then Villaggio[4] expressed the methods of Synge and Prager in modern mathematical language. Villaggio took up the problem of frame analysis, illustrating with examples how upper and lower bounds could be obtained for the deformation of simple frames. In his formulae for pointwise bounds (theorems 29.1 and 29.2) he made use of the hypersphere but not of the hypercircle.

This paper, originating in Ref. [5], builds upon the work of Prager, Synge and Villaggio. The concepts and notation of matrix structural analysis are integrated into this theoretical framework leading to a concise formulation of the bounding formulae well suited for complex, rigid-jointed frames. In order to obtain upper and lower bounds, both the displacement and force methods of structural analysis must be brought into play, thus initially doubling (approximately) the number of degrees of freedom entering into the analysis. By defining suitable subspaces, however, it is here shown that the dimensions may be greatly reduced. The resulting computations are thus made less time consuming than those of the conventional displacement or force method. The key to constructing subspaces that will yield good bounds lies in the use of "superelements" based upon the four-node rectangular

† Author to whom correspondence should be addressed.

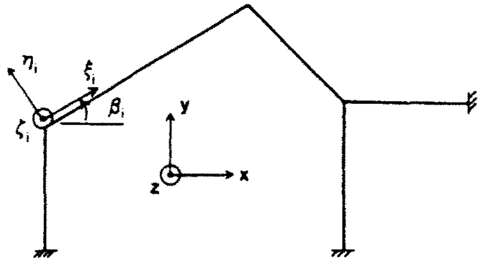

Fig. 1. Member and global axes.

finite element. It is this development, presented in Part II, that transforms the hypercircle method into a useful tool for structural analysis.

THE RIGID-JOINTED FRAME

In the idealized sense of the present treatment, a frame is a planar assemblage of straight prismatic members, loaded and deforming in its own plane. The intersection of two or more members is called a joint.

Let the members of the frame be numbered $i = 1, M$. A set of orthogonal coordinate axes is associated with each member according to the following procedure: an origin of coordinates is located at one end of the ith member and the ξ_i axis is taken along its centerline; the ζ_i axis is then directed out of the plane of the frame, and the η_i axis is taken orthogonal to the first two such that ξ_i, η_i, ζ_i form a right-handed system (Fig. 1).

THE MEMBER FIELD EQUATIONS

It is assumed that the ith member is subjected to distributed axial and normal loads $f_i^\xi(\xi_i)$ and $f_i^\eta(\xi_i)$, and a distributed couple $c_i^\zeta(\xi_i)$ (Fig. 2(a)). These functions are taken to be continuous on $[0, L_i]$, where L_i is the length of the ith member. The following notation is introduced:

$$\mathbf{f}_i = [f_i^\xi \quad f_i^\eta \quad c_i^\zeta]^T \qquad i = 1, M. \tag{1}$$

The axial force $n_i^{\xi\xi}(\xi_i)$, shear force $n_i^{\xi\eta}(\xi_i)$, and bending moment $m_i^\zeta(\xi_i)$ in the ith member are written as the vector

$$\mathbf{n}_i = [n_i^{\xi\xi} \quad n_i^{\xi\eta} \quad m_i^\zeta]^T \qquad i = 1, M. \tag{2}$$

The sign convention for these quantities is shown in Fig. 2(b). They are referred to collectively as "member forces". The member loads are related to the member forces by the equilibrium equations

$$d\mathbf{n}_i/d\xi_i + \tilde{\mathbf{n}}_i + \mathbf{f}_i = 0 \qquad i = 1, M \tag{3}$$

where

$$\tilde{\mathbf{n}}_i = [0 \quad 0 \quad n_i^{\xi\eta}]^T. \tag{4}$$

The kinematical field variables are now developed. The kinematical model of the member consists of two components: a deformable fiber which in the undeformed state lies

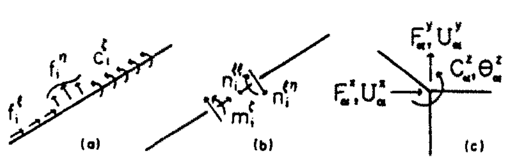

Fig. 2. Force notation: external forces, internal forces, joint forces.

along the ξ_i axis and a set of undeformable transverse fibers, one attached to each point of the longitudinal fiber. In the deformed configuration the displacement at any point along the member is described not only by the displacements $u_i^\xi(\xi_i)$, $u_i^\eta(\xi_i)$ in the ξ_i and η_i directions but also by the angle θ_i^ζ between the η_i axis and the attached fiber. These displacements are related to a set of deformations by defining the following: $e_i^{\xi\xi}(\xi_i)$ is the stretching of the deformable fiber, $e_i^{\xi\eta}(\xi_i)$ is the shear strain between this fiber and the attached fiber, and $\kappa_i^\zeta(\xi_i)$ is the portion of the curvature of the deformable fiber due to change in θ_i^ζ. The member displacements and member deformations are therefore

$$\mathbf{u}_i = [u_i^\xi \quad u_i^\eta \quad \theta_i^\zeta]^{\mathrm{T}} \qquad i = 1, M \tag{5a}$$

$$\mathbf{e}_i = [e_i^{\xi\xi} \quad e_i^{\xi\eta} \quad \kappa_i^\zeta]^{\mathrm{T}} \qquad i = 1, M. \tag{5b}$$

These quantities are related by the deformation–displacement equations

$$\mathrm{d}\mathbf{u}_i/\mathrm{d}\xi_i = \mathbf{e}_i + \tilde{\mathbf{u}}_i \quad \text{where} \quad \tilde{\mathbf{u}}_i = [0 \quad \theta_i^\zeta \quad 0]^{\mathrm{T}}, \quad i = 1, M. \tag{6, 7}$$

JOINT DATA AND JOINT-MEMBER END CONDITIONS

Let the joints of the frame be numbered $\alpha = 1, J$. Furthermore, suppose that a set of global x, y, z axes has been established (Fig. 1). Then the joint displacements and forces (Fig. 2(c)) are given by

$$\mathbf{U}_\alpha = [U_\alpha^x \quad U_\alpha^y \quad \theta_\alpha^z]^{\mathrm{T}} \qquad \alpha = 1, J \tag{8a}$$

$$\mathbf{F}_\alpha = [F_\alpha^x \quad F_\alpha^y \quad C_\alpha^z]^{\mathrm{T}} \qquad \alpha = 1, J. \tag{8b}$$

For simplicity, it is assumed that at each joint either the displacement $\mathbf{U}_\alpha$ or the force $\mathbf{F}_\alpha$ is specified. Let the joints of the first class be numbered $\alpha' = 1, J'$ and those of the second class $\alpha'' = 1, J''$. Then the joint data may be represented

$$\mathbf{U}_{\alpha'} = \hat{\mathbf{U}}_{\alpha'} \qquad \alpha' = 1, J' \tag{9a}$$

$$\mathbf{F}_{\alpha''} = \hat{\mathbf{F}}_{\alpha''} \qquad \alpha'' = 1, J'' \tag{9b}$$

where $\hat{\mathbf{U}}_{\alpha'}$ and $\hat{\mathbf{F}}_{\alpha''}$ denote the specified displacements and forces.

The rigidity of the joints requires the compatibility conditions

$$\mathbf{u}_i(L_i) = \mathbf{T}_i\mathbf{U}_{\alpha(i,L)} \quad \text{and} \quad \mathbf{u}_i(0) = \mathbf{T}_i\mathbf{U}_{\alpha(i,0)} \tag{10}$$

where $\alpha(i, L)$ and $\alpha(i, 0)$ denote the joints at which the ends $\xi_i = L_i$ and 0, respectively, are located. In eqns (10)

$$\mathbf{T}_i = \begin{bmatrix} \cos\beta_i & \sin\beta_i & 0 \\ -\sin\beta_i & \cos\beta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

where β_i is the angle between the global x axis and the ξ_i axis (Fig. 1).

The corresponding equilibrium conditions for rigid joints are

$$\mathbf{F}_\alpha = \Sigma_{i(\alpha,L)}\mathbf{T}_i^{\mathrm{T}}\mathbf{n}_i(L_i) - \Sigma_{i(\alpha,0)}\mathbf{T}_i^{\mathrm{T}}\mathbf{n}_i(0) \tag{12}$$

where $\Sigma_{i(\alpha,L)}$ and $\Sigma_{i(\alpha,0)}$ denote the sums over those member ends $\xi_i = L_i$ and 0, respectively, which lie at the αth joint.

MEMBER FORCE–DEFORMATION RELATIONS

The kinematical and statical equations of the frame have thus far been developed independently, although certain parallels may be noted. In order to complete the theory, it is necessary to introduce relations between the member force and deformation fields. These are given by

$$\mathbf{n}_i = \tilde{\mathbf{C}}_i \mathbf{e}_i \tag{13}$$

where

$$\tilde{\mathbf{C}}_i = \begin{bmatrix} (E_i A_i)^{-1} & 0 & 0 \\ 0 & (G_i \mathring{A}_i)^{-1} & 0 \\ 0 & 0 & (E_i I_i)^{-1} \end{bmatrix}. \tag{14}$$

In this matrix E_i is Young's modulus of elasticity and G_i the shear modulus for the ith member. A_i is the cross-sectional area, $\mathring{A}_i$ the "shear area", and I_i the second moment of area of the cross-section about the ζ_i axis.

Equation (13) corresponds to the well-known stress–strain relation of linear elasticity.

ELASTIC STATES

An elastic state for a given frame F is an array of member displacement, deformation, and force fields and joint displacements and force vectors. If the abbreviations

$$\mathbf{u} = \mathbf{u}_1(\xi_1), \ldots, \mathbf{u}_M(\xi_M), \quad \mathbf{e} = \mathbf{e}_1(\xi_1), \ldots, \mathbf{e}_M(\xi_M), \quad \mathbf{n} = \mathbf{n}_1(\xi_1), \ldots, \mathbf{n}_M(\xi_M)$$
$$\mathbf{U} = \mathbf{U}_1, \ldots, \mathbf{U}_J, \quad \mathbf{F} = \mathbf{F}_1, \ldots, \mathbf{F}_J \tag{15}$$

are used, then the symbolism

$$\mathbf{S} = [\mathbf{u}, \mathbf{e}, \mathbf{n}; \mathbf{U}, \mathbf{F}] \tag{16}$$

may be employed to represent an elastic state.† S will be called an admissible state if the member fields $\mathbf{u}$, $\mathbf{e}$, and $\mathbf{n}$ are all piecewise continuous. If, furthermore, the member force and deformation fields are related by eqn (13), the state S will be called an elastic state. Henceforth this term, or simply "state" will mean an admissible elastic state.

The sum of two states and multiplication by scalars is defined by

$$a\mathbf{S} + b\underline{\mathbf{S}} = [a\mathbf{u} + b\underline{\mathbf{u}}, a\mathbf{e} + b\underline{\mathbf{e}}, a\mathbf{n} + b\underline{\mathbf{n}}; a\mathbf{U} + b\underline{\mathbf{U}}, a\mathbf{F} + b\underline{\mathbf{F}}] \tag{17}$$

where

$$\underline{\mathbf{S}} = [\underline{\mathbf{u}}, \underline{\mathbf{e}}, \underline{\mathbf{n}}; \underline{\mathbf{U}}, \underline{\mathbf{F}}].$$

By virtue of this definition the set of all elastic states for a given frame is a vector space and its elements are vectors. This space is now equipped with the inner product

$$\langle \mathbf{S}, \underline{\mathbf{S}} \rangle = \sum_{i=1}^{M} \int_0^{L_i} \mathbf{e}_i \cdot \underline{\mathbf{n}}_i \, \mathrm{d}\xi_i = \sum_{i=1}^{M} \int_0^{L_i} (e_i^{\xi\xi} \underline{n}_i^{\xi} + e_i^{\xi\eta} \underline{n}_i^{\xi\eta} + \kappa_i^{\xi} \underline{m}_i^{\xi}) \, \mathrm{d}\xi_i. \tag{18}$$

It is simple to verify that

† This notation is similar to Gurtin[6].

$$\langle \mathbf{S}, a\underline{\mathbf{S}}+b\mathbf{S}' \rangle = a\langle \mathbf{S}, \underline{\mathbf{S}} \rangle + b\langle \mathbf{S}, \mathbf{S}' \rangle. \tag{19}$$

Furthermore, since $\mathbf{n}_i = \tilde{\mathbf{C}}_i\mathbf{e}_i$, $\underline{\mathbf{n}}_i = \tilde{\mathbf{C}}_i\underline{\mathbf{e}}_i$ and $\tilde{\mathbf{C}}_i$ is symmetric and positive definite, it follows that

$$\langle \mathbf{S}, \underline{\mathbf{S}} \rangle = \langle \underline{\mathbf{S}}, \mathbf{S} \rangle$$

$$\langle \mathbf{S}, \mathbf{S} \rangle \geqslant 0; \quad \langle \mathbf{S}, \mathbf{S} \rangle = 0 \quad \text{iff } \mathbf{n}_i = \mathbf{e}_i = 0, \quad i = 1, n. \tag{20}$$

Definition (18) therefore satisfies the requirements for a valid inner product (p. 54 of Shilov[7]), and the space of elastic states is now an inner product space. The norm of S is defined by

$$\|\mathbf{S}\|^2 = \langle \mathbf{S}, \mathbf{S} \rangle. \tag{21}$$

From eqn (18) it may be seen that $\|\mathbf{S}\|^2$ is twice the strain energy associated with S. The Schwarz inequality, i.e.

$$|\langle \mathbf{S}, \underline{\mathbf{S}} \rangle| \leqslant \|\mathbf{S}\| \, \|\underline{\mathbf{S}}\| \tag{22}$$

may be proved (p. 57 of Shilov[7]).

The concept of orthogonality will play a central role in the theory. Two states S, $\underline{\mathbf{S}}$ are said to be orthogonal if

$$\langle \mathbf{S}, \underline{\mathbf{S}} \rangle = 0. \tag{23}$$

KINEMATICALLY AND STATICALLY ADMISSIBLE STATES

Let a frame F composed of M members and J joints be given, along with a set of distributed member loads $\mathbf{f}_i$, $i = 1, M$ and joint data $\hat{\mathbf{U}}_{\alpha'}$, $\alpha' = 1, J'$; $\hat{\mathbf{F}}_{\alpha''}$, $\alpha'' = 1, J''$. This will be called the general frame problem.

A state S is termed kinematically admissible (KA) for this problem if:

(1) the member deformation–displacement equations, eqn (6), are satisfied everywhere (implying that the displacement fields are smooth);

(2) the joint compatibility conditions (10) are satisfied;

(3) the joint displacements are consistent with the data; i.e. eqns (9a) are satisfied.

A second class of states is defined in an analogous manner: S is statically admissible (SA) if:

(1) the member equilibrium equations, eqn (3), are satisfied;

(2) the joint equilibrium conditions, eqn (12), are satisfied;

(3) the joint forces are consistent with the data; i.e. eqns (9b) are satisfied.

With these two definitions an important result involving the inner product of a KA and SA state may be derived. Let S* be KA and S** be SA; then from definition (18)

$$\langle \mathbf{S}^*, \mathbf{S}^{**} \rangle = \sum_{i=1}^{M} \int_0^{L_i} \mathbf{e}_i^* \cdot \mathbf{n}_i^{**} \, \mathrm{d}\xi_i. \tag{24}$$

If eqns (6) and (7) are used, integration by parts and eqns (3) and (4) yield

$$\int_0^{L_i} \mathbf{e}_i^* \cdot \mathbf{n}_i^{**} \, d\xi_i = \int_0^{L_i} \mathbf{u}_i^* \cdot \mathbf{f}_i \, d\xi_i + [\mathbf{u}_i^* \cdot \mathbf{n}_i^{**}]_0^{L_i}. \tag{25}$$

But the last term on the right-hand side may be transformed using eqns (9), (10) and (12). Hence inner product (24) becomes

$$\langle \mathbf{S}^*, \mathbf{S}^{**} \rangle = \sum_{i=1}^{M} \int_0^{L_i} \mathbf{u}_i^* \cdot \mathbf{f}_i \, d\xi_i + \sum_{\alpha=1}^{J'} \mathbf{U}_\alpha \cdot \mathbf{F}_\alpha^{**} + \sum_{\alpha''=1}^{J''} \mathbf{U}_{\alpha''}^* \cdot \mathbf{F}_{\alpha''}. \tag{26}$$

This result is closely related to the principle of virtual work. It is now used to prove the following theorem.

Let the state S be the actual solution of the general frame problem, then

$$\langle \mathbf{S}^* - \mathbf{S}, \mathbf{S}^{**} - \mathbf{S} \rangle = 0 \tag{27}$$

where $\mathbf{S}^*$ and $\mathbf{S}^{**}$ are KA and SA states, respectively. To prove this, it is observed that the actual state S is, by definition, both KA and SA. Upon application of eqn (26) to the inner products $\langle \mathbf{S}^*, \mathbf{S}^{**} \rangle$, $\langle \mathbf{S}^*, \mathbf{S} \rangle$, $\langle \mathbf{S}, \mathbf{S}^{**} \rangle$, and $\langle \mathbf{S}, \mathbf{S} \rangle$, eqn (27) follows. This equation is identical to eqn (9.9) in Ref. [1].

LINEAR SUBSPACES AND HYPERPLANES

Let E denote the vector space of elastic states for a given frame. Then the notation

$$\mathbf{S} \in E \tag{28}$$

will be used to signify that S is an elastic state. From the definition, E is infinite dimensional.

A set of states $\bar{E}$ is a linear subspace of E if, for all $\mathbf{S}_1, \mathbf{S}_2 \in \bar{E}$

$$a\mathbf{S}_1 + b\mathbf{S}_2 \in \bar{E} \tag{29}$$

where a and b are arbitrary scalars. If the elements of $\bar{E}$ are regarded as vectors drawn from the origin (i.e. the **0** state), then $\bar{E}$ may be viewed as a plane passing through the origin.

It is now convenient to define an associated homogeneous problem for the general frame problem described above. This is done by letting all the member loads $\mathbf{f}_i$ vanish, and requiring that the joint data be homogeneous; i.e. no non-zero displacements or forces are specified. Let E' and E'' be the sets of states that are KA and SA, respectively, for the homogeneous problem. From the definitions it may be shown that E' and E'' are both linear subspaces of E. Furthermore, for all $\mathbf{S}' \in E'$, $\mathbf{S}'' \in E''$, eqn (26) shows that

$$\langle \mathbf{S}', \mathbf{S}'' \rangle = 0 \tag{30}$$

that is, all vectors in E' are orthogonal to all vectors in E''. These two subspaces are said to be orthogonal. E' may be termed the subspace of compatible states and E'' the subspace of residual states.

The general frame problem is now considered. Let E^* and E^{**} denote the sets of KA and SA states for this problem. These two sets are, in general, not linear subspaces of E since they do not contain the **0** state. However, if $\mathbf{S}_1^*, \mathbf{S}_2^* \in E^*$ and $\mathbf{S}_1^{**}, \mathbf{S}_2^{**} \in E^{**}$,† then it is easily seen that

$$\mathbf{S}_1^* - \mathbf{S}_2^* \in E', \quad \mathbf{S}_1^{**} - \mathbf{S}_2^{**} \in E'' \tag{31}$$

i.e. E^* and E^{**} are parallel to E' and E'', respectively. By the parallelogram law of vector addition, the difference vectors in expressions (31) may also be regarded as lying in E^* and

† This means that the KA and SA states, regarded as points, lie in E^* and E^{**}.

E^{**}. In this sense the "planes" E^* and E^{**} may be said to be orthogonal. They will be called hyperplanes. If one chooses $\mathbf{S}_2^* = \mathbf{S}_2^{**} = \mathbf{S}$ in eqns (31), then eqn (27) is recovered.

THE HYPERSPHERE AND RELATED BOUNDS

Theorem (27) may be rewritten in the form

$$\|\mathbf{S}-\mathbf{C}_0\|^2 = R_0^2 \tag{32}$$

in which

$$\mathbf{C}_0 = \tfrac{1}{2}(\mathbf{S}^*+\mathbf{S}^{**}), \quad R_0 = \tfrac{1}{2}\|\mathbf{S}^*-\mathbf{S}^{**}\|. \tag{33}$$

This means that the actual state $\mathbf{S}$ lies somewhere on a "sphere" of radius R_0 about $\mathbf{C}_0$, called a hypersphere by Prager and Synge[1]. Equation (32) may be recast in the more useful form

$$\mathbf{S} = \mathbf{C}_0+R_0\mathbf{J}, \quad \|\mathbf{J}\| = 1. \tag{34}$$

Hence

$$\|\mathbf{S}\|^2 = \|\mathbf{C}_0\|^2+2R_0\langle\mathbf{C}_0,\mathbf{J}\rangle+R_0^2. \tag{35}$$

This expression readily yields bounds on $\|\mathbf{S}\|^2$

$$(\|\mathbf{C}_0\|-R_0)^2 \leqslant \|\mathbf{S}\|^2 \leqslant (\|\mathbf{C}_0\|+R_0)^2 \tag{36}$$

when the Schwarz inequality (22) is used. These bounds correspond exactly to the geometrical idea of $\mathbf{S}$ as a point on a sphere.

In order to derive a bounding formula for a particular kinematical or statical quantity belonging to the actual state $\mathbf{S}$, the principle of virtual work is brought into play. If x represents a desired displacement, then one may construct a state $\mathbf{G}$ that is statically admissible for a set of virtual forces that does work only through the unknown displacement x. Equation (26) then gives

$$\langle\mathbf{S},\mathbf{G}\rangle = x. \tag{37}$$

Bounds on the inner product $\langle\mathbf{S},\mathbf{G}\rangle$ provide bounds on the desired displacement. An analogous procedure using virtual displacements gives a desired statical quantity in terms of an inner product. From eqn (34) there follows

$$\langle\mathbf{S},\mathbf{G}\rangle = \langle\mathbf{C}_0,\mathbf{G}\rangle+R_0\langle\mathbf{J},\mathbf{G}\rangle \tag{38}$$

and the following bounding formula is obtained:

$$\langle\mathbf{C}_0,\mathbf{G}\rangle-R_0\|\mathbf{G}\| \leqslant \langle\mathbf{S},\mathbf{G}\rangle \leqslant \langle\mathbf{C}_0,\mathbf{G}\rangle+R_0\|\mathbf{G}\|. \tag{39}$$

Formulae (36) and (38) may be found in Synge[2].

HYPERPLANES L^* AND L^{**}

Hyperplanes E^* and E^{**}, being infinite dimensional, are difficult to work in. It is convenient to define hyperplanes $L^* \subset E^*$ and $L^{**} \subset E^{**}$ that are finite dimensional. This leads to a formulation that is suitable for structural analysis.

First, the set E^* of KA states is considered. If, in addition, one requires that $S^* \in E^*$ be "almost" in E^{**} in the sense that all of the requirements for a statically admissible state are satisfied except the third requirement (i.e. the statical joint data), then a smaller set $L^* \subset E^*$ is obtained.

It is a straightforward matter to apply this definition in order to construct a state belonging to L^*. With the actual member loads placed on the structure, all the joints are held fixed except where non-zero joint displacements are prescribed (at these points the prescribed displacements are imposed). The member field equations may then be solved subject to these kinematical boundary conditions. The joint loads, computed from the joint equilibrium conditions, are the loads required to hold the joints fixed. If the state so obtained is denoted $\mathbf{S}_0^*$, then $\mathbf{S}_0^* \in L^*$ by construction. In particular, if the frame is loaded only at the joints and the kinematical data are homogeneous, then $\mathbf{S}_0^* = \mathbf{0}$.

Now the member loads are removed and the joint displacements which are not part of the kinematical data are identified by a numbering system $p = 1, 2, \ldots, n'$. The state obtained by prescribing a unit value for the pth displacement while all the others are set to zero is denoted $\mathbf{S}'_p$. A general state belonging to L^* may now be written

$$\mathbf{S}^* = \mathbf{S}_0^* + \sum_{p=1}^{n'} x_p \mathbf{S}'_p \tag{40}$$

where the x_p are scalar parameters which may be identified with the joint displacements. The second term on the right-hand side of eqn (40) defines a linear subspace L' of dimension n' (the number of kinematical degrees of freedom).

A general expression for the inner product of two states $\mathbf{S}^*$, $\underline{\mathbf{S}}^*$ belonging to E^* may be obtained. The computations leading to eqn (26) apply except that the statical data are not satisfied. If $\mathbf{S}_0^* = \underline{\mathbf{S}}_0^* = \mathbf{0}$, this inner product takes the particularly simple form

$$\langle \mathbf{S}^*, \underline{\mathbf{S}}^* \rangle = \sum_{\alpha=1}^{J} \mathbf{U}_\alpha^* \cdot \underline{\mathbf{F}}_\alpha^* = \sum_{\alpha=1}^{J} \underline{\mathbf{U}}_\alpha^* \cdot \mathbf{F}_\alpha^* \tag{41}$$

where $\mathbf{F}_\alpha^*$ and $\mathbf{U}_\alpha^*$ are the joint forces and displacements associated with $\mathbf{S}^*$.

The hyperplane $L^{**} \subset E^{**}$ is now defined. A state $\mathbf{S}^{**} \in E^{**}$ also belongs to L^{**} if it satisfies the second and third requirements for kinematic admissibility and the first requirement is satisfied "almost everywhere" in a manner that uniquely determines the displacement fields $\mathbf{u}_i(\xi_i)$ given the strain fields $\mathbf{e}_i(\xi_i)$.

A state belonging to L^{**} is found by imposing a set of n'' statical constraints which allow the member equilibrium equations to be solved for the given loading. Specifically, it is required that one or more member forces vanish at various points of the frame. The required number of such constraints is, by definition, the degree of statical indeterminacy of the structure.† Since the requirements for kinematic admissibility are "almost" satisfied, the member displacement fields and joint displacements may be determined. If the state so determined is denoted $\mathbf{S}_0^{**}$, then $\mathbf{S}_0^{**} \in L^{**}$. Furthermore, a general state in L^{**} may be expressed in the form

$$\mathbf{S}^{**} = \mathbf{S}_0^{**} + \sum_{q=1}^{n''} \sigma_q \mathbf{S}''_q. \tag{42}$$

Here the σ_q are scalar parameters that may be identified with the redundant forces and $\mathbf{S}''_q$ is the state obtained by prescribing a unit value for the qth redundant, the remaining $n'' - 1$ constraints being kept in force. The second term on the right defines a subspace L'' of dimension n''.

Since $L' \subset E'$, $L'' \subset E''$ it follows that the subspaces L' and L'' are orthogonal. This is equivalent to the condition

$$\langle \mathbf{S}'_p, \mathbf{S}''_q \rangle = 0 \qquad p = 1, n'; q = 1, n''. \tag{43}$$

Furthermore, hyperplanes $L^* \subset E^*$, $L^{**} \subset E^{**}$ are orthogonal in the sense given earlier.

† n'' is the number of statical degrees of freedom.

Henceforth $\mathbf{S}^*$ and $\mathbf{S}^{**}$ will be assumed to belong to L^* and L^{**}, respectively.

THE PRINCIPLE OF MINIMUM ENERGY DISTANCE

By construction, hyperplanes L^* and L^{**} both contain the actual state S. This state may be found by minimizing the squared distance

$$d^2 = \|\mathbf{S}^* - \mathbf{S}^{**}\|^2. \tag{44}$$

If expressions (40) and (42) are substituted into eqn (44), and the minimization is carried out by requiring that

$$\partial(d^2)/\partial x_r = 0, \quad \partial(d^2)/\partial \sigma_s = 0 \qquad r = 1, n'; s = 1, n'' \tag{45}$$

then the following two sets of linear algebraic equations are obtained for the determination of x_p and σ_q:

$$\sum_{p=1}^{n'} K_{rp} x_p - P_r = 0 \qquad r = 1, n'$$

$$\sum_{q=1}^{n''} B_{sq} \sigma_q + \Delta_s = 0 \qquad s = 1, n'' \tag{46}$$

in which

$$K_{rp} = \langle \mathbf{S}'_r, \mathbf{S}'_p \rangle = K_{pr}, \qquad P_r = \langle \mathbf{S}_0^{**} - \mathbf{S}_0^*, \mathbf{S}'_r \rangle \tag{47a}$$

$$B_{sq} = \langle \mathbf{S}''_s, \mathbf{S}''_q \rangle = B_{qs}, \qquad \Delta_s = \langle \mathbf{S}_0^{**} - \mathbf{S}_0^*, \mathbf{S}''_s \rangle. \tag{47b}$$

Equations (46) are the equations of the displacement and force methods of structural analysis. In particular if $\mathbf{S}_0^* = 0$, then P_r is simply the concentrated load corresponding to the rth joint displacement. These equations may be written in compact form

$$\mathbf{Kx} = \mathbf{P} \quad \text{and} \quad \mathbf{B}\boldsymbol{\sigma} + \boldsymbol{\Delta} = \mathbf{0}. \tag{48}$$

If this notation is used, then eqn (44) may be rewritten as

$$d^2 = \|\mathbf{S}_0^* - \mathbf{S}_0^{**}\|^2 + \mathbf{Kx}\cdot\mathbf{x} - 2\mathbf{P}\cdot\mathbf{x} + \mathbf{B}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} + 2\boldsymbol{\Delta}\cdot\boldsymbol{\sigma}. \tag{49}$$

If the solutions of eqns (48) are denoted by $\hat{\mathbf{x}}$ and $\hat{\boldsymbol{\sigma}}$, it follows that

$$\|\mathbf{S}_0^* - \mathbf{S}_0^{**}\|^2 - \mathbf{P}\cdot\hat{\mathbf{x}} + \boldsymbol{\Delta}\cdot\hat{\boldsymbol{\sigma}} = 0. \tag{50}$$

HYPERPLANES L_v^* AND L_v^{**}

Subspaces L' and L'' are of dimension n' and n'', where n' is the number of kinematical degrees of freedom and n'' the degree of statical indeterminacy of the structure. Since in practice these numbers may be quite large, it is natural to seek ways of reducing them. This may be accomplished most simply by discarding some of the vectors $\mathbf{S}'_p$ and $\mathbf{S}''_q$, leaving the sets $\mathbf{S}'_i$, $i = 1, v'$ and $\mathbf{S}''_j$, $j = 1, v''$. These define new subspaces $L'_v \subset L'$ and $L''_v \subset L''$, and the expressions

$$\mathbf{S}^* = \mathbf{S}_0^* + \sum_{i=1}^{v'} x_i \mathbf{S}'_i, \quad \mathbf{S}^{**} = \mathbf{S}_0^{**} + \sum_{j=1}^{v''} \sigma_j \mathbf{S}''_j \tag{51}$$

now represent general states in two orthogonal hyperplanes $L_v^* \subset L^*$ and $L_v^{**} \subset L^{**}$.

Minimization of the distance between these two states yields two sets of equations identical in form to eqns (48) but smaller in dimension. These equations will be termed the vertex equations. If their solutions are denoted by $\hat{\mathbf{x}}$ and $\hat{\boldsymbol{\sigma}}$, then the two states

$$\mathbf{V}^* = \mathbf{S}_0^* + \sum_{i=1}^{\nu'} \hat{x}_i \mathbf{S}_i', \quad \mathbf{V}^{**} = \mathbf{S}_0^{**} + \sum_{j=1}^{\nu''} \hat{\sigma}_j \mathbf{S}_j'' \tag{52}$$

will be called the vertices of hyperplanes L_ν^* and L_ν^{**}. These are the points of closest approach of the two hyperplanes to each other. It is now shown that the difference vector $\mathbf{V}^* - \mathbf{V}^{**}$ is orthogonal to both L_ν^* and L_ν^{**} (or, equivalently, L_ν' and L_ν''). If eqns (47), (51), and (52) are used, a straightforward computation shows that

$$\langle \mathbf{V}^* - \mathbf{V}^{**}, \mathbf{S}^* - \mathbf{V}^* \rangle = (\mathbf{K}\hat{\mathbf{x}} - \mathbf{P}) \cdot (\mathbf{x} - \hat{\mathbf{x}}) = 0 \tag{53a}$$

$$\langle \mathbf{V}^* - \mathbf{V}^{**}, \mathbf{S}^{**} - \mathbf{V}^{**} \rangle = (\mathbf{B}\hat{\boldsymbol{\sigma}} + \boldsymbol{\Delta}) \cdot (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) = 0 \tag{53b}$$

where the vertex equations, eqns (48), have been used. Now every vector in L_ν' may be given by $\mathbf{S}^* - \mathbf{V}^*$ for some $\mathbf{S}^* \in L_\nu^*$. Hence eqn (53a) implies the first of the desired orthogonalities; the second follows by a similar argument.

These orthogonalities may be used to derive additional bounding formulae on $\|\mathbf{S}\|^2$. First it is noted that since $\mathbf{S} - \mathbf{V}^{**} \in L''$ and $\mathbf{S}_i' \in L'$, there follows

$$\begin{aligned} 0 &= \langle \mathbf{S} - \mathbf{V}^{**}, \mathbf{S}_i' \rangle \qquad i = 1, \nu' \\ &= \langle \mathbf{S} - \mathbf{V}^* + \mathbf{V}^* - \mathbf{V}^{**}, \mathbf{S}_i' \rangle \\ &= \langle \mathbf{S} - \mathbf{V}^*, \mathbf{S}_i' \rangle. \end{aligned} \tag{54}$$

Similarly, it may be shown that

$$\langle \mathbf{S} - \mathbf{V}^{**}, \mathbf{S}_j'' \rangle = 0 \qquad j = 1, \nu''. \tag{55}$$

Therefore, $\mathbf{S} - \mathbf{V}^{**}$ is orthogonal to both L_ν' and L_ν''. Hence

$$\|\mathbf{S} - \mathbf{S}^*\|^2 = \|\mathbf{S} - \mathbf{V}^*\|^2 + \|\mathbf{S}^* - \mathbf{V}^*\|^2 \tag{56}$$

and since $0 \leqslant \|\mathbf{S} - \mathbf{V}^*\|^2 \leqslant \|\mathbf{V}^* - \mathbf{V}^{**}\|^2$, there follows

$$\|\mathbf{V}^* - \mathbf{S}^*\|^2 \leqslant \|\mathbf{S} - \mathbf{S}^*\|^2 \leqslant \|\mathbf{V}^* - \mathbf{S}^*\|^2 + \|\mathbf{V}^* - \mathbf{V}^{**}\|^2. \tag{57}$$

In the case where L_ν^* contains the origin, $\mathbf{S}^*$ may be set to $\mathbf{0}$. Furthermore, in this case eqn (53a) shows that $\langle \mathbf{V}^*, \mathbf{V}^* - \mathbf{V}^{**} \rangle = 0$. With these simplifications expression (57) becomes

$$\|\mathbf{V}^*\|^2 \leqslant \|\mathbf{S}\|^2 \leqslant \|\mathbf{V}^{**}\|^2 \tag{58}$$

a result given by Prager and Synge[1]. Using an exactly analogous procedure it may be shown that in the case where L_ν^{**} contains the origin inequalities (58) are reversed. The first case is equivalent to the condition $\mathbf{S}_0^* = \mathbf{0}$, and is predominant in practical applications. The following results which pertain to this case are recorded for future reference:

$$\begin{aligned} \|\mathbf{V}^*\|^2 &= \mathbf{K}\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \mathbf{P} \cdot \hat{\mathbf{x}} \\ \|\mathbf{V}^{**}\|^2 &= \|\mathbf{S}_0^{**}\|^2 + 2\boldsymbol{\Delta} \cdot \hat{\boldsymbol{\sigma}} + \mathbf{B}\hat{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{\sigma}} = \|\mathbf{S}_0^{**}\|^2 + \boldsymbol{\Delta} \cdot \hat{\boldsymbol{\sigma}}. \end{aligned} \tag{59}$$

These are obtained from eqns (47), (48), and (52).

THE HYPERCIRCLE

The hypercircle provides a means of deriving a more refined type of bounding formula. The treatment given here is based on that of Synge[2], except that an orthonormal set of vectors is not assumed to be available.

It is first recalled that $\mathbf{S}'_i$, $i = 1, v'$ and $\mathbf{S}''_j$, $j = 1, v''$ are two linearly independent sets of vectors lying in L' and L'', respectively. Now let $\mathbf{S}^*$ and $\mathbf{S}^{**}$ be any states belonging to L^*_v and L^{**}_v. Then since $\mathbf{S}-\mathbf{S}^* \in L'$ and $\mathbf{S}-\mathbf{S}^{**} \in L''$, it follows that

$$\langle \mathbf{S}-\mathbf{S}^{**}, \mathbf{S}'_i \rangle = 0, \quad i = 1, v'; \qquad \langle \mathbf{S}-\mathbf{S}^{*}, \mathbf{S}''_j \rangle = 0, \quad j = 1, v''. \tag{60}$$

These equations may be rewritten as

$$\langle \mathbf{S}, \mathbf{S}'_i \rangle = a'_i, \quad i = 1, v'; \qquad \langle \mathbf{S}, \mathbf{S}''_j \rangle = a''_j, \quad j = 1, v'' \tag{61}$$

where

$$a'_i = \langle \mathbf{S}^{**}, \mathbf{S}'_i \rangle = \langle \mathbf{S}^{**}_0, \mathbf{S}'_i \rangle \tag{62a}$$

$$a''_j = \langle \mathbf{S}^{*}, \mathbf{S}''_j \rangle = \langle \mathbf{S}^{*}_0, \mathbf{S}''_j \rangle. \tag{62b}$$

Equations (61) confine the solution $\mathbf{S}$ to a hyperplane of dimension $n'+n''-v'-v''$, or, in the terminology of Synge, a hyperplane of class $v'+v''$.

Since the states $\mathbf{S}^*$ and $\mathbf{S}^{**}$ also serve to locate $\mathbf{S}$ on a hypersphere, it may be concluded that $\mathbf{S}$ lies on the intersection of the hyperplane and the hypersphere. This intersection is given the natural name hypercircle by Synge and Prager. The equations of the hypercircle will now be developed.

The center of the hypercircle is sought in the form

$$\mathbf{C} = \mathbf{C}_0 + \sum_{i=1}^{v'} b'_i \mathbf{S}'_i + \sum_{j=1}^{v''} b''_j \mathbf{S}''_j \tag{63}$$

with the requirement that

$$\langle \mathbf{C}, \mathbf{S}'_i \rangle = a'_i, \quad i = 1, v'; \qquad \langle \mathbf{C}, \mathbf{S}''_j \rangle = a''_j, \quad j = 1, v'' \tag{64}$$

so that $\mathbf{C}$ lies in the hyperplane defined by eqns (61). In eqn (63), $\mathbf{C}_0$ is the center of the hypersphere given by eqn $(33)_1$. If the inner product of eqn (63) is taken first with $\mathbf{S}'_k$, then with $\mathbf{S}''_m$, the following sets of equations are obtained:

$$a'_k = \langle \mathbf{C}_0, \mathbf{S}'_k \rangle + \sum_{i=1}^{v'} K_{ki} b'_i \qquad k = 1, v' \tag{65a}$$

$$a''_m = \langle \mathbf{C}_0, \mathbf{S}''_m \rangle + \sum_{j=1}^{v''} B_{mj} b''_j \qquad m = 1, v'' \tag{65b}$$

where eqns (64) and (47) have been used. If eqns $(33)_1$, (47), and (62) are used the above equations may be put in the form

$$\mathbf{K}\mathbf{b}' = \tfrac{1}{2}\mathbf{c}', \quad \mathbf{B}\mathbf{b}'' = \tfrac{1}{2}\mathbf{c}'' \tag{66}$$

where

$$\mathbf{c}' = \mathbf{P} - \mathbf{K}\mathbf{x}, \quad \mathbf{c}'' = -\boldsymbol{\Delta} - \mathbf{B}\boldsymbol{\sigma}. \tag{67}$$

Equations (66) admit unique solutions for $\mathbf{b}'$ and $\mathbf{b}''$, so that point $\mathbf{C}$ given by eqn (63) is well defined. A further computation shows that eqn (63) yields the relation

$$\|\mathbf{S}-\mathbf{C}\|^2 = \|\mathbf{S}-\mathbf{C}_0\|^2 - \tfrac{1}{2}\mathbf{e}'\cdot\mathbf{b}' - \tfrac{1}{2}\mathbf{e}''\cdot\mathbf{b}''. \tag{68}$$

This may be rewritten in the form

$$\|\mathbf{S}-\mathbf{C}\|^2 = R^2 \tag{69}$$

where

$$R^2 = R_0^2 - \tfrac{1}{2}\mathbf{e}'\cdot\mathbf{b}' - \tfrac{1}{2}\mathbf{e}''\cdot\mathbf{b}'' \tag{70}$$

R_0 being the radius of the hypersphere. Equation (69) may be rewritten as

$$\mathbf{S} = \mathbf{C} + R\mathbf{J}, \quad \|\mathbf{J}\| = 1 \tag{71}$$

and from eqns (61), (64), and (71) it follows that

$$\langle \mathbf{J}, \mathbf{S}_i' \rangle = 0, \quad i = 1, \nu'; \qquad \langle \mathbf{J}, \mathbf{S}_j'' \rangle = 0, \quad j = 1, \nu''. \tag{72}$$

Equations (71) and (72) are the equations of a hypercircle with center **C** and radius R. Interpreted geometrically, eqns (72) mean that the unit vector **J** (drawn from **C**) lies in the hyperplane.

If the vertex equations, eqns (48), are satisfied, then from eqns (66) and (67) there follows

$$\mathbf{e}' = \mathbf{b}' = \mathbf{0}, \quad \mathbf{e}'' = \mathbf{b}'' = \mathbf{0}. \tag{73}$$

Hence from eqns (63) and (70)

$$\mathbf{C} = \mathbf{C}_0 = \tfrac{1}{2}(\mathbf{V}^* + \mathbf{V}^{**}), \quad R^2 = R_0^2 = \tfrac{1}{4}\|\mathbf{V}^* - \mathbf{V}^{**}\|^2. \tag{74}$$

These equations mean that, if $\mathbf{S}^* = \mathbf{V}^*$ and $\mathbf{S}^{**} = \mathbf{V}^{**}$, then the center of the hypercircle and hypersphere coincide and their radii are equal: the hypercircle is a great circle of the hypersphere.

BOUNDS USING THE HYPERCIRCLE

The hypercircle will now be used to find improved bounds on the inner product $\langle \mathbf{S}, \mathbf{G} \rangle$. First **G** is written as

$$\mathbf{G} = \hat{\mathbf{G}} + \sum_{i=1}^{\nu'} g_i' \mathbf{S}_i' + \sum_{j=1}^{\nu''} g_j'' \mathbf{S}_j'' \tag{75}$$

with the requirement that

$$\langle \hat{\mathbf{G}}, \mathbf{S}_i' \rangle = 0, \quad i = 1, \nu'; \qquad \langle \hat{\mathbf{G}}, \mathbf{S}_j'' \rangle = 0, \quad j = 1, \nu''. \tag{76}$$

This means that $\hat{\mathbf{G}}$ is the component of **G** lying in (parallel to) the hyperplane. If the inner product of eqn (75) is taken first with $\mathbf{S}_k'$, then with $\mathbf{S}_m''$, the two systems

$$\mathbf{K}\mathbf{g}' = \mathbf{z}', \quad \mathbf{B}\mathbf{g}'' = \mathbf{z}'' \tag{77}$$

are obtained for the determination of the scalar parameters g_i', g_j''. The vectors $\mathbf{z}'$ and $\mathbf{z}''$ are given by

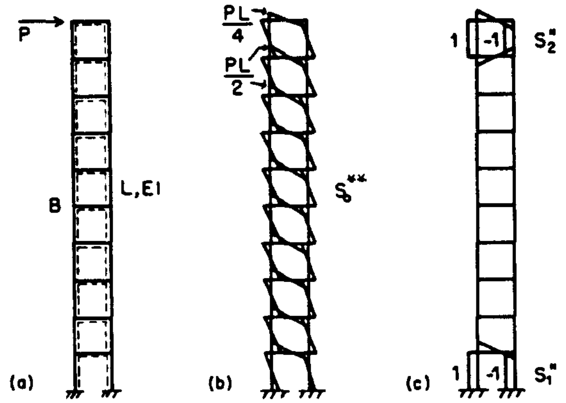

Fig. 3. Illustrative problem configuration (bending moment is positive if dotted side of member is in tension).

$$z'_k = \langle \mathbf{G}, \mathbf{S}'_k \rangle, \quad k = 1, \nu'; \qquad z''_m = \langle \mathbf{G}, \mathbf{S}''_m \rangle, \quad m = 1, \nu''. \tag{78}$$

A computation then shows that

$$\|\hat{\mathbf{G}}\|^2 = \|\mathbf{G}\|^2 - \mathbf{z}' \cdot \mathbf{g}' - \mathbf{z}'' \cdot \mathbf{g}''. \tag{79}$$

It is now possible to obtain a new bounding formula; from eqns (71)

$$\langle \mathbf{S}, \mathbf{G} \rangle = \langle \mathbf{C}, \mathbf{G} \rangle + R \langle \mathbf{J}, \mathbf{G} \rangle. \tag{80}$$

But from eqns (72) and (75) it follows that

$$\langle \mathbf{J}, \mathbf{G} \rangle = \langle \mathbf{J}, \hat{\mathbf{G}} \rangle \leqslant \|\hat{\mathbf{G}}\| \tag{81}$$

where the Schwarz inequality has been used. Hence the formula

$$\langle \mathbf{C}, \mathbf{G} \rangle - R\|\hat{\mathbf{G}}\| \leqslant \langle \mathbf{S}, \mathbf{G} \rangle \leqslant \langle \mathbf{C}, \mathbf{G} \rangle + R\|\hat{\mathbf{G}}\| \tag{82}$$

is obtained, in which $\|\hat{\mathbf{G}}\|$ is given by eqn (79). This new result represents one of the most useful products of the hypercircle method. Its application in structural analysis will be the focus of the remainder of this article. Henceforth, it will be assumed that the vertex equations, eqns (48), are satisfied. In this case, the inner product $\langle \mathbf{C}, \mathbf{G} \rangle$ may be computed as follows:

$$\begin{aligned} \langle \mathbf{C}, \mathbf{G} \rangle &= \tfrac{1}{2}(\langle \mathbf{V}^*, \mathbf{G} \rangle + \langle \mathbf{V}^{**}, \mathbf{G} \rangle) \\ &= \tfrac{1}{2}(\langle \mathbf{S}_0^*, \mathbf{G} \rangle + \mathbf{z}' \cdot \hat{\mathbf{x}} + \langle \mathbf{S}_0^{**}, \mathbf{G} \rangle + \mathbf{z}'' \cdot \hat{\boldsymbol{\sigma}} \end{aligned} \tag{83}$$

where eqns (52), (74), and (78) have been used. The radius of the hypercircle is given by eqns (49), (50), and $(74)_2$

$$4R^2 = d^2 = \|\mathbf{V}^* - \mathbf{V}^{**}\|^2 = \|\mathbf{S}_0^* - \mathbf{S}_0^{**}\|^2 - \mathbf{P} \cdot \hat{\mathbf{x}} + \boldsymbol{\Delta} \cdot \hat{\boldsymbol{\sigma}}. \tag{84}$$

For the case where $\mathbf{S}_0^* = \mathbf{0}$, the important simplification

$$4R^2 = \|\mathbf{V}^{**}\|^2 - \|\mathbf{V}^*\|^2 = \|\mathbf{S}_0^{**}\|^2 + \boldsymbol{\Delta} \cdot \hat{\boldsymbol{\sigma}} - \mathbf{P} \cdot \hat{\mathbf{x}} \tag{85}$$

occurs. The "hat" on $\hat{\mathbf{x}}$ and $\hat{\boldsymbol{\sigma}}$ may be suppressed when it is implied by the context.

AN ILLUSTRATIVE EXAMPLE

To illustrate the application of these bounding formulae to a specific problem, the structure and loading shown in Fig. 3(a) is considered. Bounds are wanted for the lateral

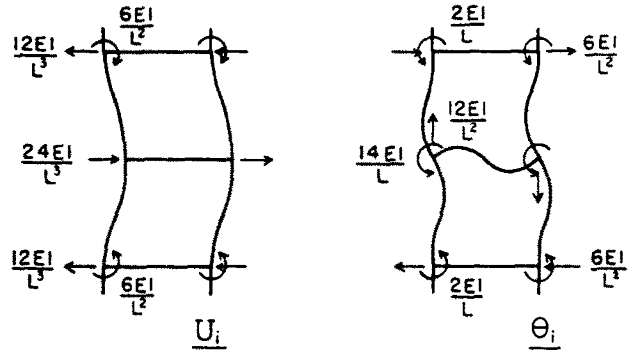

Fig. 4. Joint displacements and forces associated with states U_i and θ_i.

displacement (drift) at the roof and fifth story levels. For didactic reasons it is desirable to carry out the computations by hand; therefore deformations due to axial load and shear forces are ignored.

First the state $\mathbf{S}_0^{**}$ is constructed by inserting hinges at the midpoints of all members. The bending moment diagram thus obtained is shown in Fig. 3(b). It is then assumed that the actual state may be approximated by a statically admissible state of the form $\mathbf{S}^{**} = \mathbf{S}_0^{**} + \sigma_1 \mathbf{S}_1'' + \sigma_2 \mathbf{S}_2''$. This is simply eqn $(51)_2$ with $r'' = 2$. In the above equation $\mathbf{S}_1''$ and $\mathbf{S}_2''$ are the residual† states shown in Fig. 3(c). To obtain the "best" values of σ_1 and σ_2, the vertex equations, eqn $(48)_2$, are applied. Explicitly

$$\sigma_1 \langle \mathbf{S}_1'', \mathbf{S}_1'' \rangle + \sigma_2 \langle \mathbf{S}_1'', \mathbf{S}_2'' \rangle = \langle \mathbf{S}_0^* - \mathbf{S}_0^{**}, \mathbf{S}_1'' \rangle$$

$$\sigma_1 \langle \mathbf{S}_2'', \mathbf{S}_1'' \rangle + \sigma_2 \langle \mathbf{S}_2'', \mathbf{S}_2'' \rangle = \langle \mathbf{S}_0^* - \mathbf{S}_0^{**}, \mathbf{S}_2'' \rangle.$$

The inner products are computed from definition (18). Clearly, $\mathbf{S}_0^* = \mathbf{0}$ and $\langle \mathbf{S}_1'', \mathbf{S}_2'' \rangle = B_{1,2} = 0$. The remaining inner products are

$$B_{1,1} = (7/3)(L/EI), \quad B_{2,2} = (8/3)(L/EI)$$

$$\Delta_1 = (PL^2)/(6EI), \quad \Delta_2 = -(PL^2)/(12EI).$$

Then

$$\sigma_1 = -\Delta_1/B_{1,1} = -PL/14, \quad \sigma_2 = -\Delta_2/B_{2,2} = PL/32.$$

A computation also shows that

$$\|\mathbf{S}_0^{**}\|^2 = (19/16)(P^2L^3/EI)$$

so that

$$\|\mathbf{V}^{**}\|^2 = \|\mathbf{S}_0^{**}\|^2 + \Delta \cdot \sigma = 1.1730(P^2L^3/EI).$$

Next a kinematically admissible state $\mathbf{S}^*$ which approximates $\mathbf{S}$ must be found. This is done by first introducing the two compatible states $\mathbf{U}_i$ and $\boldsymbol{\theta}_i$, where $\mathbf{U}_i$ is the state obtained by imposing a unit horizontal displacement on both joints of the ith story and $\boldsymbol{\theta}_i$ is the state obtained by imposing a unit rotation on the same two joints. The joint displacements and forces associated with these states are shown in Fig. 4.

For this structure and loading, it is known that the displacements tend to increase linearly and the rotations tend to a constant value. This suggests the following linear combinations of $\mathbf{U}_i$ and $\boldsymbol{\theta}_i$:

† Henceforth "residual state" implies $\mathbf{S}'' \in L''$; "compatible state" implies $\mathbf{S}' \in L'$.

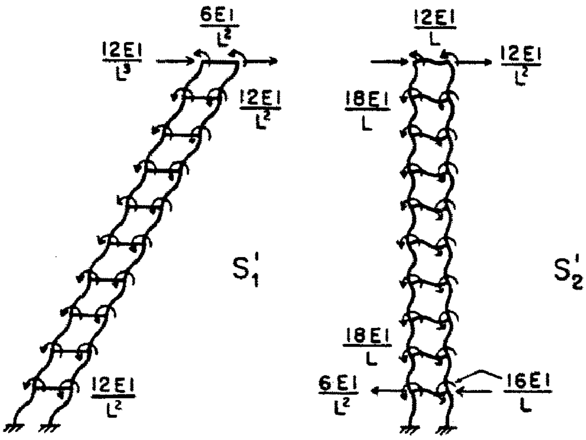

Fig. 5. Joint displacements and forces associated with states S'_1 and S'_2 (vertical forces not shown in S'_2).

$$\mathbf{S}'_1 = \mathbf{U}_1+2\mathbf{U}_2+3\mathbf{U}_3+\cdots+10\mathbf{U}_{10}, \quad \mathbf{S}'_2 = \boldsymbol{\theta}_1+\boldsymbol{\theta}_2+\boldsymbol{\theta}_3+\cdots+\boldsymbol{\theta}_{10}.$$

The joint displacement and forces associated with these states are shown in Fig. 5. A kinematically admissible state of the form

$$\mathbf{S}^* = x_1\mathbf{S}'_1+x_2\mathbf{S}'_2$$

is now sought. The vertex equations, eqn $(48)_1$, are applied, with the relevant inner products computed from eqn (41)

$$K_{1,1} = 240(EI/L^3), \quad K_{1,2} = K_{2,1} = 228(EI/L^2), \quad K_{2,2} = 344(EI/L)$$

$$P_1 = 10P, \qquad P_2 = 0$$

$$\begin{bmatrix} 240 & 228L \\ 228L & 344L^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 10PL^3/EI \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

which has the solution

$$x_1 = (430/3822)(PL^3/EI), \quad x_2 = -(285/3822)(PL^2/EI).$$

Hence

$$\|\mathbf{V}^*\|^2 = \mathbf{P}\cdot\mathbf{x} = 1.1251P^2L^3/EI$$

and the bounding formula (58) gives

$$1.1251P^2L^3/EI \leqslant \|\mathbf{S}\|^2 \leqslant 1.1730P^2L^3/EI.$$

Since the single load, P, is applied at the top of the frame, the above bound on the strain energy gives immediate bounds on the drift. Given the relative simplicity of the computations these bounds are remarkably close. They could be improved as much as desired, naturally at the expense of additional computation.

Expression (82) may be used to find bounds on the deflection at the fifth story level. The application of a unit load to the hinged structure at this point, B, as shown in Fig.

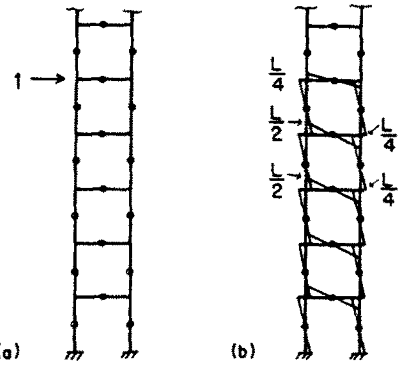

Fig. 6. Loading and bending moment diagram for state G.

6(a), results in the bending moment diagram shown in Fig. 6(b). If the associated state is denoted **G**, the desired displacement is given by

$$u_B = \langle \mathbf{G}, \mathbf{S} \rangle.$$

The following inner products are then computed:

$$\|\mathbf{G}\|^2 = 9L^3/16EI, \quad \langle \mathbf{G}, \mathbf{S}_0^{**} \rangle = 7PL^3/12EI$$

$$z'_1 = \langle \mathbf{G}, \mathbf{S}'_1 \rangle = 5, \quad z''_1 = \langle \mathbf{G}, \mathbf{S}''_1 \rangle = L^2/6EI, \quad z'_2 = z''_2 = 0.$$

The solution of eqns (77) then gives

$$g'_1 = 86(5L^3/7644EI), \quad g'_2 = -57(5L^3/7644EI), \quad g''_1 = L/14, \quad g''_2 = 0.$$

The computations may then be completed as follows: from eqn (83)

$$\langle \mathbf{G}, \mathbf{C} \rangle = \tfrac{1}{2}(\mathbf{z}' \cdot \mathbf{x} + \langle \mathbf{G}, \mathbf{S}_0^{**} \rangle + \mathbf{z}'' \cdot \boldsymbol{\sigma})$$

$$= (PL^3/2EI)\,[(2150/3822) + (7/12) - (1/84)] = 0.5670(PL^3/EI)$$

and from eqn (85)

$$R^2 = \tfrac{1}{4}(\|\mathbf{V}^{**}\|^2 - \|\mathbf{V}\|^2) = 0.01198(P^2L^3/EI).$$

Finally, from eqn (79)

$$\|\hat{\mathbf{G}}\|^2 = \|\mathbf{G}\|^2 - \mathbf{z}' \cdot \mathbf{g}' - \mathbf{z}'' \cdot \mathbf{g}'' = (L^3/EI)\,[(9/16) - (2150/7644) - (1/84)] = 0.2693(L^3/EI).$$

If these values are substituted into the bounding formula (82), the following bounds are obtained:

$$u_B \geq \langle \mathbf{G}, \mathbf{C} \rangle - R\|\hat{\mathbf{G}}\| = 0.5102(PL^3/EI)$$

$$u_B \leq \langle \mathbf{G}, \mathbf{C} \rangle + R\|\hat{\mathbf{G}}\| = 0.6238(PL^3/EI).$$

This completes the example.

GENERAL LINEAR SUBSPACES

Linear subspaces L'_r and L''_r, created simply by discarding some of the vectors $\mathbf{S}'_p$ and $\mathbf{S}''_q$, are too restrictive to be useful for practical cases. As seen in the preceding example, it may be necessary to employ more general types of subspaces. Such a treatment is now presented.

Let x_p and σ_q be the "old" kinematical and statical parameters, respectively, and let $\underline{x}_i$ ($i = 1, v' \leqslant n'$) and $\underline{\sigma}_j$ ($j = 1, v'' \leqslant n''$) be the "new" sets of parameters. Then the linear transformations

$$x_p = \sum_{i=1}^{v'} \Gamma_{pi}\underline{x}_i, \quad p = 1, n'; \qquad \sigma_q = \sum_{j=1}^{v''} \Omega_{qj}\underline{\sigma}_j, \quad q = 1, n'' \tag{86}$$

define linear subspaces $L'_\Gamma \subset L'$ and $L''_\Omega \subset L''$. The vertices of the associated hyperplanes L^*_Γ and L^{**}_Ω are located by solving the vertex equations

$$\underline{\mathbf{K}}\underline{\mathbf{x}} = \underline{\mathbf{P}}, \quad \underline{\mathbf{B}}\underline{\boldsymbol{\sigma}} + \underline{\boldsymbol{\Delta}} = 0. \tag{87}$$

The transformations giving $\underline{\mathbf{K}}, \underline{\mathbf{P}}$ and $\underline{\mathbf{B}}, \underline{\boldsymbol{\Delta}}$ are obtained by appealing to a familiar invariance argument. The squared distance, eqn (49), should remain invariant under the transformations, i.e.

$$\|\mathbf{S}^*_0 - \mathbf{S}^{**}_0\|^2 + \underline{\mathbf{K}}\underline{\mathbf{x}}\cdot\underline{\mathbf{x}} - 2\underline{\mathbf{P}}\cdot\underline{\mathbf{x}} + \underline{\mathbf{B}}\underline{\boldsymbol{\sigma}}\cdot\underline{\boldsymbol{\sigma}} + 2\underline{\boldsymbol{\Delta}}\cdot\underline{\boldsymbol{\sigma}}$$
$$= \|\mathbf{S}^*_0 - \mathbf{S}^{**}_0\|^2 + \mathbf{K}\mathbf{x}\cdot\mathbf{x} - 2\mathbf{P}\cdot\mathbf{x} + \mathbf{B}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} + 2\boldsymbol{\Delta}\cdot\boldsymbol{\sigma}. \tag{88}$$

Transformations (86) are written in compact form and substituted in the right-hand side of eqn (88). Since the resulting equation must hold for all $\underline{\mathbf{x}}, \underline{\boldsymbol{\sigma}}$, it follows that

$$\underline{\mathbf{K}} = \boldsymbol{\Gamma}^T\mathbf{K}\boldsymbol{\Gamma}, \quad \underline{\mathbf{P}} = \boldsymbol{\Gamma}^T\mathbf{P}; \qquad \underline{\mathbf{B}} = \boldsymbol{\Omega}^T\mathbf{B}\boldsymbol{\Omega}, \quad \underline{\boldsymbol{\Delta}} = \boldsymbol{\Omega}^T\boldsymbol{\Delta}. \tag{89}$$

The form of these expressions is well known. The other vectors entering into the hypercircle method are obtained similarly; eqns (77) become

$$\underline{\mathbf{K}}\underline{\mathbf{g}}' = \underline{\mathbf{z}}', \quad \underline{\mathbf{B}}\underline{\mathbf{g}}'' = \underline{\mathbf{z}}'' \tag{90}$$

in which

$$\underline{\mathbf{z}}' = \boldsymbol{\Gamma}^T\mathbf{z}', \quad \underline{\mathbf{z}}'' = \boldsymbol{\Omega}^T\mathbf{z}''. \tag{91}$$

Once eqns (87) and (90) have been solved, the quantities appearing in the bounding formula (82) may be computed with the unbarred vectors replaced by their barred counterparts. Equations (59), (79), and (83) become

$$\|\mathbf{V}^*\|^2 = \underline{\mathbf{P}}\cdot\underline{\mathbf{x}}, \quad \|\mathbf{V}^{**}\|^2 = \|\mathbf{S}^{**}_0\|^2 + \underline{\boldsymbol{\Delta}}\cdot\underline{\boldsymbol{\sigma}} \tag{92a}$$

$$\|\hat{\mathbf{G}}\|^2 = \|\mathbf{G}\|^2 - \underline{\mathbf{z}}'\cdot\underline{\mathbf{g}}' - \underline{\mathbf{z}}''\cdot\underline{\mathbf{g}}'' \tag{92b}$$

$$\langle \mathbf{C}, \mathbf{G}\rangle = \tfrac{1}{2}(\langle \mathbf{S}^*_0, \mathbf{G}\rangle + \underline{\mathbf{z}}'\cdot\underline{\mathbf{x}} + \langle \mathbf{S}^{**}_0, \mathbf{G}\rangle + \underline{\mathbf{z}}''\cdot\underline{\boldsymbol{\sigma}}). \tag{92c}$$

CONCLUSION

The theoretical basis for the application of function space methods to the structural analysis of rigid-jointed planar frames has been developed. The formulae permit computation of bounds on deformation and stress. These bounds may be made as narrow as desired, though, understandably, additional accuracy entails additional computational effort. The theory is illustrated by a didactic example. Practical application of the method to large structures having many members is, however, deferred to Part II.

Acknowledgement—One of us (C.A.N.) acknowledges with thanks the support of the Department of Civil Engineering, University of Manitoba, Canada.

REFERENCES

1. W. Prager and J. L. Synge, Approximations in elasticity based on the concept of function space. *Q. Appl. Math.* **5**, 241–269 (1947).
2. J. L. Synge, *The Hypercircle in Mathematical Physics*. Cambridge University Press, Cambridge (1957).
3. W. Prager, The extremum principles of the mathematical theory of elasticity and their use in stress analysis. *Bull. Univ. Wash. Engng Station* No. 119 (1950).
4. P. Villaggio, *Qualitative Methods in Elasticity*. Noordhoff, Leyden, The Netherlands (1977).
5. C. A. Nelson, Drift bounds for tall buildings by function space methods. Thesis submitted to the Graduate School, University of Minnesota, in partial fulfillment of the requirements for the Ph.D. degree (May 1985).
6. M. E. Gurtin, The linear theory of elasticity. In *Handbuch der Physik*, Vol. VI a/2. Springer, Berlin (1972).
7. G. Shilov, *Elementary Functional Analysis* (translated from Russian). MIT Press, Cambridge, Massachusetts (1974).

APPENDIX. NOTATION

$\mathbf{B}$	flexibility matrix; see eqn (47b)
$\check{\mathbf{C}}$	elasticity matrix; see eqn (14)
$\mathbf{C}_0$	vector locating center of hypersphere
$\mathbf{C}$	vector locating center of hypercircle
d	"distance" between two elastic states; see eqn (44)
E	vector space of elastic states for a given frame
$\mathbf{e}_i$	deformation vector for member i, $[e_i^{\xi} \quad e_i^{\eta} \quad \kappa_i^{\zeta}]^T$
$\mathbf{F}_\alpha$	external forces, joint α (see Fig. 2(c)), $[F_\alpha^x \quad F_\alpha^y \quad C_\alpha^z]^T$
$\mathbf{f}_i$	external loads on member i (see Fig. 2(a)), $[f_i^{\xi} \quad f_i^{\eta} \quad c_i^{\zeta}]^T$
$\mathbf{G}$	statically admissible state for virtual loading
$\mathbf{g}$	vectors defined by eqns (77)
$\mathbf{J}$	unit vector; see eqns (34)
$\mathbf{K}$	stiffness matrix; see eqn (47a)
L	finite-dimensional subspace of E
$\mathbf{n}_i$	axial force, shear, bending moment (see Fig. 2(b)), $[n_i^{\xi} \quad n_i^{\eta} \quad m_i^{\zeta}]^T$
$\mathbf{P}$	concentrated joint force vector; see eqn (48)
R_0	radius of hypersphere
R	radius of hypercircle
$\mathbf{S}$	elastic state vector; see eqn (16), $[\mathbf{u}, \mathbf{e}, \mathbf{n}; \mathbf{U}, \mathbf{F}]$
$\mathbf{T}$	member rotation vector; see eqn (11)
$\mathbf{U}_\alpha$	displacement vector for joint α (see Fig. 2(c)), $[U_\alpha^x \quad U_\alpha^y \quad \theta_\alpha^z]^T$
$\mathbf{u}_i$	displacement vector for member i, $[u_i^{\xi} \quad u_i^{\eta} \quad \theta_i^{\zeta}]^T$
$\mathbf{V}$	vertex of L
$\boldsymbol{\iota}$	vector defined by eqns (78)
$\mathbf{\Lambda}$	vectors defined by eqn (47b)
ξ_i, η_i, ζ_i	Cartesian axes associated with member i (see Fig. 1).